

A decomposition theorem for $\{\text{ISK4}, \text{wheel}\}$ -free trigraphs

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Abstract

An *ISK4* in a graph G is an induced subgraph of G that is isomorphic to a subdivision of K_4 (the complete graph on four vertices). A *wheel* is a graph that consists of a chordless cycle, together with a vertex that has at least three neighbors in the cycle. A graph is $\{\text{ISK4}, \text{wheel}\}$ -free if it has no *ISK4* and does not contain a wheel as an induced subgraph. A “trigraph” is a generalization of a graph in which some pairs of vertices have “undetermined” adjacency. We prove a decomposition theorem for $\{\text{ISK4}, \text{wheel}\}$ -free trigraphs. Our proof closely follows the proof of a decomposition theorem for *ISK4*-free graphs due to Lévêque, Maffray, and Trotignon (On graphs with no induced subdivision of K_4 . J. Combin. Theory Ser. B, 102(4):924–947, 2012).

1 Introduction

All graphs in this manuscript are finite and simple. If H and G are graphs, we say that G is H -free if G does not contain (an isomorphic copy of) H

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as an induced subgraph. If \mathcal{H} is a family of graphs, a graph G is said to be \mathcal{H} -free if G is H -free for all $H \in \mathcal{H}$.

An ISK_4 in a graph G is an induced subgraph of G that is isomorphic to a subdivision of K_4 (the complete graph on four vertices). A *wheel* is a graph that consists of a chordless cycle, together with a vertex that has at least three neighbors in the cycle. L  v  que, Maffray, and Trotignon [3] proved a decomposition theorem for ISK_4 -free graphs and then derived a decomposition theorem for $\{ISK_4, \text{wheel}\}$ -free graphs as a corollary. Here, we are interested in a class that generalizes the class of $\{ISK_4, \text{wheel}\}$ -free graphs, namely, the class of $\{ISK_4, \text{wheel}\}$ -free “trigraphs.” Trigraphs (originally introduced by Chudnovsky [1, 2] in the context of Berge graphs) are a generalization of graphs in which certain pairs of vertices may have “undetermined” adjacency (one can think of such pairs as “optional edges”). Every graph can be thought of as a trigraph: a graph is simply a trigraph with no “optional edges.” Trigraphs and related concepts are formally defined in Section 3.

We now wish to state the decomposition theorem for $\{ISK_4, \text{wheel}\}$ -free graphs from [3], but we first need a few definitions. A graph is *series-parallel* if it does not contain any subdivision of K_4 as a (not necessarily induced) subgraph. The *line graph* of a graph H , denoted by $L(H)$, is the graph whose vertices are the edges of H , and in which two vertices (i.e., edges of H) are adjacent if they share an endpoint in H . A graph is *chordless* if all its cycles are induced.

If H is an induced subgraph of a graph G and $v \in V(G) \setminus V(H)$, then the *attachment* of v over H in G is the set of all neighbors of v in $V(H)$. If S is either a set of vertices or an induced subgraph of $G \setminus V(H)$, then the *attachment* of S over H in G is the set of all vertices of H that are adjacent to at least one vertex of S . A *square* is a cycle of length four. If S is an induced square of a graph G , say with vertices a_1, a_2, a_3, a_4 (with subscripts understood to be modulo 4) that appear in that order in S , then a *long link* of S in G is an induced path P of $G \setminus V(S)$ that contains at least one edge, and satisfies the property that there is an index $i \in \{1, 2, 3, 4\}$ such that the attachment of one endpoint of P over S is $\{a_i, a_{i+1}\}$, the attachment of the other endpoint of P over S is $\{a_{i+2}, a_{i+3}\}$, and no interior vertex of P has a neighbor in S . A *long rich square* is a graph G that contains an induced square S (called a *central square* of G) such that $G \setminus V(S)$ contains at least two components, and all such components are long links of S in G .

A *clique* of a graph G is a (possibly empty) set of pairwise adjacent vertices of G , and a *stable set* of G is a (possibly empty) set of pairwise non-adjacent vertices of G . A *cutset* of a graph G is a (possibly empty) set of vertices whose deletion from G yields a disconnected graph. A *clique-cutset* of a graph G is clique of G that is also a cutset of G . (Note that if G is a disconnected graph, then \emptyset is a clique-cutset of G .) A *proper 2-cutset* of a graph G is a cutset $\{a, b\}$ of G that is a stable set of size two such that

$V(G) \setminus \{a, b\}$ can be partitioned into two non-empty sets X and Y so that there is no edge between X and Y and neither $G[X \cup \{a, b\}]$ nor $G[Y \cup \{a, b\}]$ is a path between a and b .

We are now ready to state the decomposition theorem for $\{\text{ISK4}, \text{wheel}\}$ -free graphs from [3] (this is Theorem 1.2 from [3]).

Theorem 1.1. [3] *Let G be an $\{\text{ISK4}, \text{wheel}\}$ -free graph. Then at least one of the following holds:*

- G is series-parallel;
- G is the line graph of a chordless graph with maximum degree at most three;
- G is a complete bipartite graph;
- G is a long rich square;
- G has a clique-cutset or a proper 2-cutset.

We remark, however, that the fourth outcome of Theorem 1.1 (that is, the outcome that G is a long rich square) is in fact unnecessary. To see this, suppose that G is a long rich square, and let S be a central square of G . Choose any two-edge path S' of the square S , and choose two components, call them P_1 and P_2 , of $G \setminus V(S)$. By the definition of a long rich square, P_1 and P_2 are long links of S in G , and we see that $W = G[V(S') \cup V(P_1) \cup V(P_2)]$ is a wheel. Indeed, if x is the “central” vertex of S' (i.e., the unique vertex of the two-edge path S' that is adjacent to the other two vertices of S'), then $W \setminus x$ is a chordless cycle, and x has four neighbors in this cycle. Thus, long rich squares are not wheel-free (and consequently, they are not $\{\text{ISK4}, \text{wheel}\}$ -free). This observation allows us to strengthen Theorem 1.1 as follows.

Theorem 1.2. *Let G be an $\{\text{ISK4}, \text{wheel}\}$ -free graph. Then at least one of the following holds:*

- G is series-parallel;
- G is the line graph of a chordless graph with maximum degree at most three;
- G is a complete bipartite graph;
- G has a clique-cutset or a proper 2-cutset.

Our goal in this manuscript is to prove a trigraph version of Theorem 1.2. In Section 2, we state a few lemmas about “cyclically 3-connected graphs” proven in [3]. In Section 3, we define trigraphs and introduce some basic

trigraph terminology. Finally, in Section 4, we prove our decomposition theorem for $\{\text{ISK4}, \text{wheel}\}$ -free trigraphs (Theorem 4.1), which is very similar to Theorem 1.2. We prove Theorem 4.1 by imitating the proof of the decomposition theorem for ISK4-free graphs from [3]. Interestingly, the fact that we work with trigraphs rather than graphs does not substantially complicate the proof. On the other hand, the fact that we restrict ourselves to the wheel-free case significantly simplifies the argument (indeed, some of the most difficult parts of the proof of the theorem for ISK4-free graphs from [3] involve ISK4-free graphs that contain induced wheels).

2 Cyclically 3-connected graphs

In this section, we state a few lemmas proven in [3], but first, we need some definitions. Given a graph G , a vertex $u \in V(G)$, and a set $X \subseteq V(G) \setminus \{u\}$, we say that u is *complete* (respectively: *anti-complete*) to X in G provided that u is adjacent (respectively: non-adjacent) to every vertex of X in G . Given a graph G and disjoint sets $X, Y \subseteq V(G)$, we say that X is *complete* (respectively: *anti-complete*) to Y in G provided that every vertex of X is complete (respectively: anti-complete) to Y in G . A *separation* of a graph H is a pair (A, B) of subsets of $V(H)$ such that $V(H) = A \cup B$, and $A \setminus B$ is anti-complete to $B \setminus A$. A separation (A, B) of H is *proper* if both $A \setminus B$ and $B \setminus A$ are non-empty. A *k-separation* of H is a separation (A, B) of H such that $|A \cap B| \leq k$. A separation (A, B) is *cyclic* if both $H[A]$ and $H[B]$ have cycles. A graph H is *cyclically 3-connected* if it is 2-connected, is not a cycle, and admits no cyclic 2-separation. Note that a cyclic 2-separation of any graph is proper. A *theta* is any subdivision of the complete bipartite graph $K_{2,3}$. As usual, if H_1 and H_2 are graphs, we denote by $H_1 \cup H_2$ the graph whose vertex set is $V(H_1) \cup V(H_2)$ and whose edge set is $E(H_1) \cup E(H_2)$.

The *length* of a path is the number of edges that it contains. A *branch vertex* in a graph G is a vertex of degree at least three. A *branch* in a graph G is an induced path P of length at least one whose endpoints are branch vertices of G and all of whose interior vertices are of degree two in G .

We now state the lemmas from [3] that we need. The five lemmas below are Lemmas 4.3, 4.5, 4.6, 4.7, and 4.8 from [3], respectively.

Lemma 2.1. [3] *Let H be a cyclically 3-connected graph, let a and b be two branch vertices of H , and let P_1, P_2 , and P_3 be three induced paths of H whose ends are a and b . Then one of the following holds:*

- P_1, P_2, P_3 are branches of H of length at least two and $H = P_1 \cup P_2 \cup P_3$ (so H is a theta);
- there exist distinct indices $i, j \in \{1, 2, 3\}$ and a path S of H with one end in the interior of P_i and the other end in the interior of P_j , such

that no interior vertex of S belongs to $V(P_1 \cup P_2 \cup P_3)$, and such that $P_1 \cup P_2 \cup P_3 \cup S$ is a subdivision of K_4 .

Lemma 2.2. [3] *A graph is cyclically 3-connected if and only if it is either a theta or a subdivision of a 3-connected graph.*

Lemma 2.3. [3] *Let H be a cyclically 3-connected graph, and let a and b be two distinct vertices of H . If no branch of H contains both a and b , then $H' = (V(H), E(H) \cup \{ab\})$ is a cyclically 3-connected graph and every graph obtained from H' by subdividing ab is cyclically 3-connected.*

Lemma 2.4. [3] *Let H be a cyclically 3-connected graph, let Z be a cycle of H , and let a, b, c, d be four pairwise distinct vertices of Z that lie in this order on Z and satisfy $ab, cd \in E(Z)$. Let P be the subpath of Z from a to d that does not contain b and c , and let Q be the subpath of Z from b to c that does not contain a and d . Suppose that there exist distinct branches F_{ab} and F_{cd} of H such that $ab \in E(F_{ab})$ and $cd \in E(F_{cd})$. Then there is a path R of H such that one endpoint of R belongs to P and the other to Q , such that no interior vertex of R belongs to Z , and such that R is not from a to b or from c to d .*

Lemma 2.5. [3] *Let H be a subdivision of a 3-connected graph. Let C be a cycle of H and e an edge of H such that C and e are edgewise disjoint. Then some subgraph of H that contains C and e is a subdivision of K_4 .*

3 Trigraphs

Given a set S , we denote by $\binom{S}{2}$ the set of all subsets of S of size two. A *trigraph* is an ordered pair $G = (V(G), \theta_G)$, where $V(G)$ is a finite set, called the *vertex set* of G (members of $V(G)$ are called *vertices* of G), and $\theta_G : \binom{V(G)}{2} \rightarrow \{-1, 0, 1\}$ is a function, called the *adjacency function* of G . The *null* trigraph is the trigraph whose vertex set is empty; a *non-null* trigraph is any trigraph whose vertex set is non-empty. If G is a trigraph and $u, v \in V(G)$ are distinct, we usually write uv instead of $\{u, v\}$ (note that this means that $uv = vu$), and furthermore:

- if $\theta_G(uv) = 1$, we say that uv is a *strongly adjacent pair* of G , or that u and v are *strongly adjacent* in G , or that u is *strongly adjacent* to v in G , or that v is a *strong neighbor* of u in G , or that u and v are the *endpoints of a strongly adjacent pair* of G ;
- if $\theta_G(uv) = 0$, we say that uv is a *semi-adjacent pair* of G , or that u and v are *semi-adjacent* in G , or that u is *semi-adjacent* to v in G , or that v is a *weak neighbor* of u in G , or that u and v are the *endpoints of a semi-adjacent pair* of G ;

- if $\theta_G(uv) = -1$, we say that uv is a *strongly anti-adjacent pair* of G , or that u and v are *strongly anti-adjacent* in G , or that u is *strongly anti-adjacent* to v in G , or that v is a *strong anti-neighbor* of u in G , or that u and v are the *endpoints of a strongly anti-adjacent pair* of G ;
- if $\theta_G(uv) \geq 0$, we say that uv is an *adjacent pair* of G , or that u and v are *adjacent* in G , or that u is *adjacent* to v in G , or that v is a *neighbor* of u in G , or that u and v are the *endpoints of an adjacent pair* of G ;
- if $\theta_G(uv) \leq 0$, we say that uv is an *anti-adjacent pair* of G , or that u and v are *anti-adjacent* in G , or that u is *anti-adjacent* to v in G , or that v is an *anti-neighbor* of u in G , or that u and v are the *endpoints of an anti-adjacent pair* of G .

Note that a semi-adjacent pair is simultaneously an adjacent pair and an anti-adjacent pair. One can think of strongly adjacent pairs as “edges,” of strongly anti-adjacent pairs as “non-edges,” and of semi-adjacent pairs as “optional edges.” Clearly, any graph can be thought of as a trigraph: a graph is simply a trigraph with no semi-adjacent pairs, that is, the adjacency function of a graph G is a mapping from $\binom{V(G)}{2}$ to the set $\{-1, 1\}$.

Given a trigraph G , a vertex $u \in V(G)$, and a set $X \subseteq V(G) \setminus \{u\}$, we say that u is *complete* (respectively: *strongly complete*, *anti-complete*, *strongly anti-complete*) to X in G provided that u is adjacent (respectively: strongly adjacent, anti-adjacent, strongly anti-adjacent) to every vertex of X in G . Given a trigraph G and disjoint sets $X, Y \subseteq V(G)$, we say that X is *complete* (respectively: *strongly complete*, *anti-complete*, *strongly anti-complete*) to Y in G provided that every vertex of X is complete (respectively: strongly complete, anti-complete, strongly anti-complete) to Y in G .

Isomorphism between trigraphs is defined in the natural way. The *complement* of a trigraph $G = (V(G), \theta_G)$ is the trigraph $\overline{G} = (V(\overline{G}), \theta_{\overline{G}})$ such that $V(\overline{G}) = V(G)$ and $\theta_{\overline{G}} = -\theta_G$. Thus, \overline{G} is obtained from G by turning all strongly adjacent pairs of G into strongly anti-adjacent pairs, and turning all strongly anti-adjacent pairs of G into strongly adjacent pairs; semi-adjacent pairs of G remain semi-adjacent in \overline{G} .

Given trigraphs G and \tilde{G} , we say that \tilde{G} is a *semi-realization* of G provided that $V(\tilde{G}) = V(G)$ and for all distinct $u, v \in V(\tilde{G}) = V(G)$, we have that if $\theta_G(uv) = 1$ then $\theta_{\tilde{G}}(uv) = 1$, and if $\theta_G(uv) = -1$ then $\theta_{\tilde{G}}(uv) = -1$. Thus, a semi-realization of a trigraph G is any trigraph that can be obtained from G by “deciding” the adjacency of some semi-adjacent pairs of G , that is, by possibly turning some semi-adjacent pairs of G into strongly adjacent or strongly anti-adjacent pairs. (In particular, every trigraph is a semi-realization of itself.) A *realization* of a trigraph G is a graph that is a semi-realization of G . Thus, a realization of a trigraph G is any graph that can be obtained by “deciding” the adjacency of all semi-adjacent pairs

of G , that is, by turning each semi-adjacent pair of G into an edge or a non-edge. Clearly, if a trigraph G has m semi-adjacent pairs, then G has 3^m semi-realizations and 2^m realizations. The *full realization* of a trigraph G is the graph obtained from G by turning all semi-adjacent pairs of G into strongly adjacent pairs (i.e., edges), and the *null realization* of G is the graph obtained from G by turning all semi-adjacent pairs of G into strongly anti-adjacent pairs (i.e., non-edges).

A *clique* (respectively: *strong clique*, *stable set*, *strongly stable set*) of a trigraph G is a set of pairwise adjacent (respectively: strongly adjacent, anti-adjacent, strongly anti-adjacent) vertices of G . Note that any subset of $V(G)$ of size at most one is both a strong clique and a strongly stable set of G . Note also that if $S \subseteq V(G)$, then S is a (strong) clique of G if and only if S is a (strongly) stable set of \overline{G} . Note furthermore that if K is a strong clique and S is a stable set of G , then $|K \cap S| \leq 1$; similarly, if K is a clique and S is a strongly stable set of G , then $|K \cap S| \leq 1$. However, if K is a clique and S is a stable set of G , then we are only guaranteed that vertices in $K \cap S$ are pairwise semi-adjacent to each other, and it is possible that $|K \cap S| \geq 2$. A *triangle* (respectively: *strong triangle*) is a clique (respectively: strong clique) of size three.

Given a trigraph G and a set $X \subseteq V(G)$, the *subtrigraph of G induced by X* , denoted by $G[X]$, is the trigraph with vertex set X and adjacency function $\theta_G \upharpoonright \binom{X}{2}$, where for a function $f : A \rightarrow B$ and a set $A' \subseteq A$, we denote by $f \upharpoonright A'$ the restriction of f to A' . Given $v_1, \dots, v_p \in V(G)$, we often write $G[v_1, \dots, v_p]$ instead of $G[\{v_1, \dots, v_p\}]$. If $H = G[X]$ for some $X \subseteq V(G)$, we also say that H is an *induced subtrigraph* of G ; when convenient, we relax this definition and say that H is an induced subtrigraph of G provided that there is some set $X \subseteq V(G)$ such that H is isomorphic to $G[X]$. Further, for a trigraph G and a set $X \subseteq V(G)$, we set $G \setminus X = G[V(G) \setminus X]$; for $v \in V(G)$, we often write $G \setminus v$ instead of $G \setminus \{v\}$. The trigraph $G \setminus X$ (respectively: $G \setminus v$) is called the subtrigraph of G obtained by *deleting X* (respectively: by *deleting v*).

If H is a graph, we say that a trigraph G is an *H -trigraph* if some realization of G is (isomorphic to) H . Further, if H is a graph and G a trigraph, we say that G is *H -free* provided that all realizations of G are H -free (equivalently: provided that no induced subtrigraph of G is an H -trigraph). If \mathcal{H} is a family of graphs, we say that a trigraph G is *\mathcal{H} -free* provided that G is H -free for all graphs $H \in \mathcal{H}$. In particular, a trigraph is *ISK4-free* (respectively: *wheel-free*, *{ISK4, wheel}-free*) if all its realizations are ISK4-free (respectively: wheel-free, {ISK4, wheel}-free).

Given a graph H , a trigraph G , vertices $v_1, \dots, v_p \in V(G)$, sets $X_1, \dots, X_q \subseteq V(G)$, and induced subtrigraphs G_1, \dots, G_r of G (with $p, q, r \geq 0$), we say that $v_1, \dots, v_p, X_1, \dots, X_q, G_1, \dots, G_r$ *induce an H -trigraph in G* provided that $G[\{v_1, \dots, v_p\} \cup X_1 \cup \dots \cup X_q \cup V(G_1) \cup \dots \cup V(G_r)]$ is an H -trigraph.

A trigraph is *connected* if its full realization is a connected graph. A trigraph is *disconnected* if it is not connected. A *component* of a non-null trigraph G is any (inclusion-wise) vertex-maximal connected induced subtrigraph of G . Clearly, if H is an induced subtrigraph of a non-null trigraph G , then we have that H is a component of G if and only if the full realization of H is a component of the full realization of G .

A trigraph is a *path* if at least one of its realizations is a path. A trigraph is a *narrow path* if its full realization is a path. We often denote a path P by $v_0 - v_1 - \dots - v_k$ (with $k \geq 0$), where v_0, v_1, \dots, v_k are the vertices of P that appear in that order on some realization \tilde{P} of P such that \tilde{P} is a path. The *endpoints* of a narrow path are the endpoints of its full realization; if a and b are the endpoints of a narrow path P , then we also say that P is a *narrow (a, b) -path*, and that P is a narrow path *between* a and b . If P is a narrow path and $a, b \in V(P)$, we denote by $a - P - b$ the minimal connected induced subtrigraph of P that contains both a and b (clearly, $a - P - b$ is a narrow path between a and b). The *interior vertices* of a narrow path are the interior vertices of its full realization. The *interior* of a narrow path is the set of all interior vertices of that narrow path. The *length* of a narrow path is one less than the number of vertices that it contains. (In other words, the length of a narrow path is the number of edges that its full realization has.) A *path* (respectively: *narrow path*) in a trigraph G is an induced subtrigraph P of G such that P is a path (respectively: narrow path).

Note that if G is a connected trigraph, then for all vertices $a, b \in V(G)$, there exists a narrow path between a and b in G . (To see this, consider the full realization \tilde{G} of G . \tilde{G} is connected, and so there is a path in \tilde{G} between a and b ; let P be a shortest such path in \tilde{G} . The minimality of P guarantees that P is an induced path of \tilde{G} . But now $G[V(P)]$ is a narrow path of G between a and b .)

A *hole* of a trigraph G is an induced subtrigraph C of G such that some realization of C is a chordless cycle of length at least four. We often denote a hole C of G by $v_0 - v_1 - \dots - v_{k-1} - v_0$ (with $k \geq 4$ and indices in \mathbb{Z}_k), where v_0, v_1, \dots, v_{k-1} are the vertices of C that appear in that order in some realization \tilde{C} of C such that \tilde{C} is a chordless cycle of length at least four.

The *degree* of a vertex v in a trigraph G , denoted by $\deg_G(v)$, is the number of neighbors that v has in G . A *branch vertex* in a trigraph G is a vertex of degree at least three. A *branch* in a trigraph G is a narrow path P between two distinct branch vertices of G such that no interior vertex of P is a branch vertex. A *flat branch* in a trigraph G is a branch P of G such that no adjacent pair of P is contained in a triangle of G . (Note that every branch of length at least two is flat.) If a and b are the endpoints of a branch (respectively: flat branch) P of G , then we also say that P is an *(a, b) -branch* (respectively: *(a, b) -flat branch*) of G .

A *cutset* of a trigraph G is a (possibly empty) set $C \subseteq V(G)$ such that $G \setminus C$ is disconnected. A *cut-partition* of a trigraph G is a partition (A, B, C)

of $V(G)$ such that A and B are non-empty (C may possibly be empty), and A is strongly anti-complete to B . Note that if (A, B, C) is a cut-partition of G , then C is a cutset of G . Conversely, every cutset of G induces at least one cut-partition of G . A *clique-cutset* of a trigraph G is a (possibly empty) strong clique C of G such that $G \setminus C$ is disconnected. A *cut-vertex* of a trigraph G is a vertex $v \in V(G)$ such that $G \setminus v$ is disconnected. Note that if v is a cut-vertex of G , then $\{v\}$ is a clique-cutset of G . A *stable 2-cutset* of a trigraph G is cutset of G that is a stable set of size two. We remark that if C is a cutset of a trigraph G such that $|C| \leq 2$, then C is either a clique-cutset or a stable 2-cutset of G .

A trigraph is *series-parallel* if its full realization is series-parallel (equivalently: if all its realizations are series-parallel).

A *complete bipartite trigraph* is a trigraph whose vertex set can be partitioned into two strongly stable sets that are strongly complete to each other. A *bipartition* of a complete bipartite trigraph G is a partition (A, B) of $V(G)$ such that A and B are strongly stable sets, strongly complete to each other. A complete bipartite trigraph G is *thick* if both sets of its bipartition contain at least three vertices. A trigraph is a *strong $K_{3,3}$* if its full realization is a $K_{3,3}$ and it contains no semi-adjacent pairs. Clearly, a strong $K_{3,3}$ is a thick complete bipartite trigraph.

A *line trigraph* of a graph H is a trigraph G whose full realization is (isomorphic to) the line graph of H , and all of whose triangles are strong. A trigraph G is said to be a *line trigraph* provided there is a graph H such that G is a line trigraph of H .

4 A decomposition theorem for {ISK4,wheel}-free trigraphs

The main result of this manuscript is the following decomposition theorem for {ISK4,wheel}-free graphs.

Theorem 4.1. *Let G be an {ISK4,wheel}-free trigraph. Then G satisfies at least one of the following:*

- G is a series-parallel trigraph;
- G is a complete bipartite trigraph;
- G is a line trigraph;
- G admits a clique-cutset;
- G admits a stable 2-cutset.

The remainder of this manuscript is devoted to proving Theorem 4.1. We begin with a simple but useful proposition.

Proposition 4.2. *Let G be a non-null connected trigraph that is not a narrow path. Then G contains three distinct vertices such that the deletion of any one of them yields a connected trigraph.*

Proof. Let \tilde{G} be the full realization of G . It suffices to show that \tilde{G} contains three distinct vertices such that the deletion of any one of them from \tilde{G} yields a connected graph. Since G is connected and not a narrow path, we know that \tilde{G} is connected and not a path. If \tilde{G} is a cycle, then the deletion of any one of its vertices yields a connected graph, and since any cycle has at least three vertices, we are done. So assume that \tilde{G} is not a cycle. Then \tilde{G} contains a vertex x of degree at least three. Let T be a breadth-first search spanning tree of \tilde{G} rooted at x . Then $\deg_T(x) = \deg_{\tilde{G}}(x) \geq 3$. Since T is a tree that contains a vertex of degree at least three, we know that T contains at least three leaves. But clearly, for any leaf v of T , the graph $\tilde{G} \setminus v$ (and hence also the trigraph $G \setminus v$) is connected. \square

4.1 Diamonds in wheel-free trigraphs

The *diamond* is the graph obtained by deleting one edge from K_4 . Equivalently, the *diamond* is the unique (up to isomorphism) graph on four vertices and five edges.

Proposition 4.3. *Let G be a $\{K_4, \text{wheel}\}$ -free trigraph. Then either G is diamond-free, or G admits a clique-cutset, or G admits a stable 2-cutset.*

Proof. We assume that G is not diamond-free, and that it contains no cut-vertices, for otherwise we are done. Using the fact that G is not diamond-free, we fix an inclusion-wise maximal clique C of size at least two such that at least two vertices in $V(G) \setminus C$ are complete to C . Let A be the set of all vertices in $V(G) \setminus C$ that are complete to C in G . By construction, $|C|, |A| \geq 2$, and since G is K_4 -free, we deduce that $|C| = 2$ (set $C = \{c_1, c_2\}$), and that A is a strongly stable set. Now, we claim that C is a cutset of G . Suppose otherwise. Then there exists a narrow path in $G \setminus C$ between two distinct vertices in A ; among all such narrow paths, choose a narrow path P of minimum length, and let $a, a' \in A$ be the two endpoints of P . By the minimality of P , we know that $V(P) \cap A = \{a, a'\}$. Furthermore, since A is a strongly stable set, we know that P is of length at least two.

If at least one vertex of C , say c_1 , is anti-complete to $V(P) \setminus \{a, a'\}$, then $a - P - a' - c_1 - a$ is a hole in G , and c_2 has at least three neighbors (namely, a, a' , and c_1) in it, contrary to the fact that G is wheel-free. Thus, neither c_1 nor c_2 is anti-complete to $V(P) \setminus \{a, a'\}$. Further, since $V(P) \cap A = \{a, a'\}$, we know that no interior vertex of P is adjacent to both c_1 and c_2 . Now, let p_1 be the (unique) interior vertex of P such that p_1 has a neighbor in C , and such that the interior of the narrow path $a - P - p_1$ is strongly anti-complete to C . By symmetry, we may assume that p_1 is adjacent to c_1 (and

therefore strongly anti-adjacent to c_2). Next, let p_2 be the (unique) interior vertex of P such that p_2 is adjacent to c_2 , and c_2 is strongly anti-complete to the interior of $a - P - p_2$. Note that p_1 is an interior vertex of the narrow path $a - P - p_2$. Now $a - P - p_2 - c_2 - a$ is a hole in G , and c_1 has at least three neighbors (namely, a , p_1 , and c_2) in it, contrary to the fact that G is wheel-free. This proves that C is a cutset of G . Since C is a cutset of size two of G , we see that C is either a clique-cutset or a stable 2-cutset of G . \square

4.2 Attachment to a prism

A *prism* is a trigraph K that consists of two vertex-disjoint strong triangles, call them $\{x, y, z\}$ and $\{x', y', z'\}$, an (x, x') -flat branch P_x , a (y, y') -flat branch P_y , and a (z, z') -flat branch P_z , such that xy and $x'y'$ are the only adjacent pairs between P_x and P_y , and similarly for the other two pairs of flat branches. Sets $\{x, y, z\}$, $\{x', y', z'\}$, $V(P_x)$, $V(P_y)$, and $V(P_z)$ are called the *pieces* of the prism. We call $\{x, y, z\}$ and $\{x', y', z'\}$ the *triangle pieces* of the prism, and we call $V(P_x)$, $V(P_y)$, and $V(P_z)$ the *branch pieces* of the prism. When convenient, we refer to the flat branches P_x , P_y , and P_z of K (rather than to the sets $V(P_x)$, $V(P_y)$, and $V(P_z)$) as the branch pieces of K .

In what follows, we will often consider a prism K (with P_x , P_y , and P_z as in the definition of a prism) that is an induced subtrigraph of a trigraph G . We remark that in this case, P_x , P_y , and P_z need only be flat branches of K , and not necessarily of G .

Let G be a trigraph, and let H and C be induced subtrigraphs of G on disjoint vertex sets. The *attachment* of C over H in G is the set of all vertices of H that have a neighbor in C . Furthermore,

- C is of *type triangle* with respect to H in G provided that the attachment of C over H is contained in a strong triangle of H ;
- C is of *type branch* with respect to H in G provided that the attachment of C over H is contained in a flat branch of H .
- C is an *augmenting path* of H in G provided all the following are satisfied:
 - C is a narrow path of length at least one;
 - the interior of C is strongly anti-complete to $V(H)$ in G ;
 - if a and b are the endpoints of C , then there exist two distinct flat branches of H , call them F_a and F_b , such that for each $x \in \{a, b\}$, x has exactly two neighbors (call them x_1 and x_2) in H , $\{x, x_1, x_2\}$ is a strong triangle of G , and $x_1, x_2 \in V(F_x)$ (note that this implies that x_1x_2 is a strongly adjacent pair of F_x).

If G is a trigraph, H an induced subtrigraph of G , and $v \in V(G) \setminus V(H)$, then the *attachment* of v over H in G is the set of all vertices of H that are adjacent to v ; furthermore, we say that v is of *type triangle* (respectively: of *type branch*) with respect to H in G provided that $G[v]$ is of type triangle (respectively: of type branch) with respect to H in G .

Proposition 4.4. *Let G be an $\{ISK4, \text{wheel}, \text{diamond}\}$ -free trigraph, let K be an induced subtrigraph of G such that K is a prism, and let $v \in V(G) \setminus V(K)$. Then v has at most two neighbors in K , and furthermore, v is of type branch with respect to K .*

Proof. Let $\{x, y, z\}$, $\{x', y', z'\}$, P_x , P_y , and P_z be the pieces of the prism K , as in the definition of a prism. Note that v has at most one neighbor in $\{x, y, z\}$, for otherwise, $G[v, x, y, z]$ would be either a diamond-trigraph or a K_4 -trigraph, contrary to the fact that G is $\{ISK4, \text{diamond}\}$ -free. Similarly, v has at most one neighbor in $\{x', y', z'\}$. Suppose first that v has neighbors in each of P_x , P_y , and P_z . Let x^L be the neighbor of v in P_x such that v is strongly anti-complete to the interior of the narrow path $x - P_x - x^L$, and let y^L and z^L be chosen analogously. Then v , $x - P_x - x^L$, $y - P_y - y^L$, and $z - P_z - z^L$ induce an $ISK4$ -trigraph in G (here, we use the fact that v has at most one neighbor in $\{x', y', z'\}$), contrary to the fact that G is $ISK4$ -free. Thus, v has neighbors in at most two of P_x , P_y , and P_z , and by symmetry, we may assume that v is strongly anti-complete to P_z . If v has more than two neighbors in K (and therefore in $V(P_x) \cup V(P_y)$, since v is strongly anti-complete to P_z), then v , P_x , and P_y induce a wheel-trigraph in G , which is a contradiction. Thus, v has at most two neighbors in K .

It remains to show that v is of type branch. We may assume that v has a unique neighbor (call it v_x) in P_x , and a unique neighbor (call it v_y) in P_y , for otherwise, v is of type branch with respect to K , and we are done. Since v has at most one neighbor in $\{x, y, z\}$ and at most one neighbor in $\{x', y', z'\}$, we know that either $v_x \neq x'$ and $v_y \neq y$, or $v_x \neq x$ and $v_y \neq y'$; by symmetry, we may assume that $v_x \neq x'$ and $v_y \neq y$. But now v , $x - P_x - v_x$, P_y , and P_z induce an $ISK4$ -trigraph in G , contrary to the fact that G is $ISK4$ -free. \square

Lemma 4.5. *Let G be an $\{ISK4, \text{wheel}, \text{diamond}\}$ -free trigraph, let K be an induced subtrigraph of G such that K is a prism, and let P be an inclusion-wise minimal connected induced subtrigraph of $G \setminus V(K)$ such that P is neither of type branch nor of type triangle with respect to K . Then P is an augmenting path of K .*

Proof. Let $\{x, y, z\}$, $\{x', y', z'\}$, P_x , P_y , and P_z be the pieces of the prism K , as in the definition of a prism. By Proposition 4.4, P contains more than one vertex. Let us show that P is a narrow path. Suppose otherwise; then Proposition 4.2 guarantees that P contains three distinct vertices, call

them a , b , and c , such that the deletion of any one of them from P yields a connected trigraph. By the minimality of P , each of $P \setminus a$, $P \setminus b$, and $P \setminus c$ is of type branch or triangle with respect to K . Then for each $v \in \{a, b, c\}$, there exists a piece X_v of the prism K such that the attachment of $P \setminus v$ over K is contained in X_v , and v has a neighbor \tilde{v} in $V(K) \setminus X_v$. Then $\tilde{a} \in (X_b \cap X_c) \setminus X_a$, $\tilde{b} \in (X_a \cap X_c) \setminus X_b$, and $\tilde{c} \in (X_a \cap X_b) \setminus X_c$. Thus, X_a , X_b , and X_c are pairwise distinct and pairwise intersect. But no three pieces of K have this property. This proves that P is a narrow path.

Let p and p' be the endpoints of the narrow path P ; since P has at least two vertices, we know that $p \neq p'$. By the minimality of P , we know that there exist distinct pieces A and A' of the prism K such that the attachment of $P \setminus p'$ over K is included in A , and the attachment of $P \setminus p$ over K is included in A' . Thus, the attachment of p over K is included in A , the attachment of p' over K is included in A' , and the attachment of every interior vertex of P over K is included in $A \cap A'$. Since A and A' are distinct pieces of K , we see that $|A \cap A'| \leq 1$. Furthermore, by the minimality of P , the attachment of p over K is non-empty, as is the attachment of p' over K .

Let us show that the interior of the narrow path P is strongly anti-complete to K . Suppose otherwise. Then $A \cap A' \neq \emptyset$, and it follows that one of A and A' is a triangle and the other a flat branch of K ; by symmetry, we may assume that $A = \{x, y, z\}$ and $A' = V(P_x)$. Thus, every interior vertex of P is either strongly anti-complete to K , or has exactly one neighbor (namely x) in K ; furthermore, at least one interior vertex of P is adjacent to x (because the attachment of the interior of P over K is non-empty). By Proposition 4.4, p is of type branch, and it therefore has at most one neighbor in the triangle $\{x, y, z\}$; since the attachment of p over K is included in $\{x, y, z\}$, it follows that p has exactly one neighbor in K (and that neighbor belongs to the set $\{x, y, z\}$). If x is the unique neighbor of p in K , then P is of type branch with respect to K , which is a contradiction. So by symmetry, we may assume that y is the unique neighbor of p in K . Since P is not of type triangle, we know that p' has a neighbor in $V(P_x) \setminus \{x\}$; let $v' \in V(P_x) \setminus \{x\}$ be the neighbor of p' such that p' is strongly anti-complete to the interior of $v' - P_x - x'$. Then P , $v' - P_x - x'$, P_y , and P_z induce an ISK4-trigraph in G , which is a contradiction. This proves that the interior of P is strongly anti-complete to K .

Now, by Proposition 4.4, we know that both p and p' are of type branch; since the interior of P is strongly anti-complete to K , and since P is not of type branch with respect to K , we may assume by symmetry that the attachment of p over K is included in P_x , while the attachment of p' over K is included in P_y . Recall that the attachment of p over K is non-empty, as is the attachment of p' over K . Further, by Proposition 4.4, each of p and p' has at most two neighbors in K . If the attachment of P over K contains at least three vertices, and at most three vertices in the attachment of P over K have a strong neighbor in $\{p, p'\}$, then it is easy to see that P_x , P_y , and

P induce an ISK4-trigraph in G , contrary to the fact that G is ISK4-free. Thus, either each of p, p' has a unique neighbor in K , or each of p, p' has exactly two neighbors (both of them strong) in K .

Suppose first that each of p and p' has a unique neighbor in K . Let w be the unique neighbor of p in K , and let w' be the unique neighbor of p' in K ; by construction, $w \in V(P_x)$ and $w' \in V(P_y)$. Since P is not of type triangle with respect to K , we know that $\{w, w'\} \neq \{x, y\}$ and $\{w, w'\} \neq \{x', y'\}$. Clearly then, either $w \neq x$ and $w' \neq y'$, or $w \neq x'$ and $w' \neq y$; by symmetry, we may assume that $w \neq x'$ and $w' \neq y$. But then $P, x - P_x - w, P_y$, and P_z induce an ISK4-trigraph in G , contrary to the fact that G is ISK4-free.

We now have that each of p and p' has exactly two neighbors in K , and that each of those neighbors is strong. Let w_1, w_2 be the two neighbors of p in K , and let w'_1, w'_2 be the two neighbors of p' in K . Clearly, $w_1, w_2 \in V(P_x)$ and $w'_1, w'_2 \in V(P_y)$; by symmetry, we may assume that w_2 does not lie on the narrow path $x - P_x - w_1$, and that w'_2 does not lie on the narrow path $y - P_y - w'_1$. Now, if $w_1 w_2$ and $w'_1 w'_2$ are strongly adjacent pairs, then P is an augmenting path for K in G , and we are done. So assume that at least one of $w_1 w_2$ and $w'_1 w'_2$ is an anti-adjacent pair; by symmetry, we may assume that $w_1 w_2$ is an anti-adjacent pair. But now $x - P_x - w_1, w_2 - P_x - x', P_y$, and P induce an ISK4-trigraph in G , which is a contradiction. This completes the argument. \square

4.3 Line trigraphs

We remind the reader that a *line trigraph* of a graph H is a trigraph G whose full realization is the line graph of H , and all of whose triangles are strong. We also remind the reader that a graph is *chordless* if all its cycles are induced. A *chordless subdivision* of a graph H is any chordless graph that can be obtained by possibly subdividing the edges of H . We observe that if G is a wheel-free trigraph that is a line trigraph of a graph H , then H is chordless. (In fact, it is not hard to see that if G is a line trigraph of a graph H , then G is wheel-free if and only if H is chordless. However, we do not use this stronger fact in what follows.)

Proposition 4.6. *Let G be an $\{ISK4, \text{wheel}, \text{diamond}\}$ -free trigraph, let K be an induced subtrigraph of G such that K is a line trigraph of a chordless subdivision H of K_4 , and let $v \in V(G) \setminus V(K)$. Then v has at most two neighbors in K , and furthermore, v is of type branch with respect to K .*

Proof. Since H is a chordless subdivision of K_4 , we know that each edge of K_4 is subdivided at least once to obtain H . Let a, b, c , and d be the four vertices of H of degree three. For each $x \in \{a, b, c, d\}$, the three edges of H incident with x form a strong triangle of K , and we label this strong triangle T_x . In K , for all distinct $x, y \in \{a, b, c, d\}$, there is a unique narrow path (which we call P_{xy}) such that one endpoint of this narrow path belongs to

T_x , the other endpoint belongs to T_y , and no interior vertex of this narrow path belongs to any one of the four strong triangles T_a, T_b, T_c , and T_d . For all distinct $x, y \in \{a, b, c, d\}$, we have that P_{xy} is of length at least one (because H is obtained by subdividing each edge of K_4 at least once) and that $P_{xy} = P_{yx}$; furthermore, the six narrow paths ($P_{ab}, P_{ac}, P_{ad}, P_{bc}, P_{bd}$, and P_{cd}) are vertex-disjoint. Finally, we have that $V(K) = V(P_{ab}) \cup V(P_{ac}) \cup V(P_{ad}) \cup V(P_{bc}) \cup V(P_{bd}) \cup V(P_{cd})$, and for all adjacent pairs uu' of K , we have that either there exists some $x \in \{a, b, c, d\}$ such that $u, u' \in T_x$, or there exist distinct $x, y \in \{a, b, c, d\}$ such that $u, u' \in V(P_{xy})$.

For all distinct $x, y \in \{a, b, c, d\}$, set $K_{xy} = G \setminus V(P_{xy})$, and note that K_{xy} is a prism, and so by Proposition 4.4, v has at most two neighbors in K_{xy} and is of type branch with respect to K_{xy} . Therefore, for all distinct $x, y \in \{a, b, c, d\}$, v is strongly anti-complete to some flat branch of K_{xy} . Consequently, there exist some distinct $x, y \in \{a, b, c, d\}$ such that v is strongly anti-complete to P_{xy} ; by symmetry, we may assume that v is strongly anti-complete to P_{ab} . Since (by Proposition 4.4), v has at most two neighbors in K_{ab} , it follows that v has at most two neighbors in K . It remains to show that v is of type branch with respect to K . If v has at most one neighbor in K_{ab} (and therefore in K), then this is immediate. So assume that v has exactly two neighbors (call them v_1 and v_2) in K_{ab} (and therefore in K). By Proposition 4.4, v is of type branch with respect to K_{ab} . Consequently, we have that either $v_1, v_2 \in V(P_{ac}) \cup V(P_{ad})$, or $v_1, v_2 \in V(P_{bc}) \cup V(P_{bd})$, or $v_1, v_2 \in V(P_{cd})$. Let us assume that v is not of type branch with respect to K . Then by symmetry, we may assume that $v_1 \in V(P_{bc})$ and $v_2 \in V(P_{bd})$. But now v is not of type branch with respect to the prism K_{cd} , contrary to Proposition 4.4. This completes the argument. \square

Lemma 4.7. *Let G be an $\{ISK_4, \text{wheel}, \text{diamond}\}$ -free trigraph, let K be an induced subtrigraph of G such that K is a line trigraph of a chordless subdivision H of K_4 , and let P be an inclusion-wise minimal connected induced subtrigraph of $G \setminus V(K)$ such that P is neither of type branch nor of type triangle with respect to K . Then P is an augmenting path for K .*

Proof. Let vertices a, b, c, d of H , strong triangles T_a, T_b, T_c, T_d of K , narrow paths $P_{ab}, P_{ac}, P_{ad}, P_{bc}, P_{bd}, P_{cd}$ of K , and prisms $K_{ab}, K_{ac}, K_{ad}, K_{bc}, K_{bd}, K_{cd}$ be as in the proof of Proposition 4.6. We call $T_a, T_b, T_c, T_d, V(P_{ab}), V(P_{ac}), V(P_{ad}), V(P_{bc}), V(P_{bd}), V(P_{cd})$ the *pieces* of K . Set $T_a = \{v_{a,b}, v_{a,c}, v_{a,d}\}$, $T_b = \{v_{b,a}, v_{b,c}, v_{b,d}\}$, $T_c = \{v_{c,a}, v_{c,b}, v_{c,d}\}$, and $T_d = \{v_{d,a}, v_{d,b}, v_{d,c}\}$, so that for all distinct $x, y \in \{a, b, c, d\}$, the endpoints of P_{xy} are $v_{x,y}$ and $v_{y,x}$.

We first show that P is a narrow path. If not, then by Proposition 4.2, there exist distinct vertices $v_1, v_2, v_3 \in V(P)$ such that for all $i \in \{1, 2, 3\}$, $P \setminus v_i$ is connected. By the minimality of P , there exist pieces X_1, X_2, X_3 of K such that for all $i \in \{1, 2, 3\}$, the attachment of $P \setminus v_i$ over K is included

in X_i , and v_i has a neighbor (call it y_i) in $K \setminus X_i$. Then $y_1 \in (X_2 \cap X_3) \setminus X_1$, $y_2 \in (X_1 \cap X_3) \setminus X_2$, and $y_3 \in (X_1 \cap X_2) \setminus X_3$. Thus, X_1 , X_2 , and X_3 are pairwise distinct, and they pairwise intersect. But this is impossible because one easily sees by inspection that no three pieces of K have this property. Thus, P is a narrow path. Furthermore, since P is not of type branch with respect to K , Proposition 4.6 guarantees that P is of length at least one. Let p and p' be the endpoints of P .

By the minimality of P , we know that there exists a piece A of P such that the attachment of $P \setminus p'$ is included in A , and there exists a piece A' of P such that the attachment of $P \setminus p$ over K is included in A' . Then the attachment of P over K is included in $A \cup A'$, and the attachment of the interior of P over K is included in $A \cap A'$. Further, since P is neither of type branch nor of type triangle with respect to K , we know that $A \neq A'$.

Let us show that the interior of P is strongly anti-complete to K . Since the attachment of the interior of P over K is included in $A \cap A'$, we may assume that $A \cap A' \neq \emptyset$. Thus, A and A' are distinct pieces of K that have a non-empty intersection. Then there exist distinct $x, y \in \{a, b, c, d\}$ such that one of A and A' is $V(P_{xy})$ and the other one is T_x ; by symmetry, we may assume that $A = V(P_{ab})$ and $A' = T_a$, so that the attachment of P over K is included in $V(P_{ab}) \cup T_a$; in particular then, P is strongly anti-complete to P_{cd} , and the attachment of the interior of P over K is included in $A \cap A' = \{v_{ab}\}$. Now consider the prism K_{cd} . We know that the attachment of P over K_{cd} is the same as the attachment of P over K , and it is easy to see that P is a minimal connected induced subtrigraph of $G \setminus V(K)$ that is neither of type branch nor of type triangle with respect to K_{cd} . Then by Lemma 4.5, P is an augmenting path of K_{cd} , and in particular, the interior of P is strongly anti-complete to K_{cd} , and therefore (since P is strongly anti-complete to P_{cd}), the interior of P is strongly anti-complete to K , as we had claimed.

We now have that the attachment of p over K is included in A , the attachment of p' over K is included in A' , and the interior of P is strongly anti-complete to K . By Proposition 4.6, we know that both p and p' are of type branch, and so we may assume that A and A' are both vertex sets of flat branches of K . By symmetry, we may assume that $A = V(P_{ab})$ and $A' \in \{V(P_{ac}), V(P_{cd})\}$. Since P is not of type branch with respect to K , we know that each of p and p' has a neighbor in K . Furthermore, if $A' = V(P_{ac})$, then either p has a neighbor in $V(P_{ab}) \setminus \{v_{a,b}\}$ or p' has a neighbor in $V(P_{ac}) \setminus \{v_{a,c}\}$, for otherwise, the attachment of P over K is included in T_a , contrary to the fact that P is not of type triangle with respect to K .

Now, if $A' = V(P_{ac})$, then set $K' = K_{cd}$, and if $A' = V(P_{cd})$, then set $K' = K_{ac}$. It is easy to see that P is a minimal connected induced subtrigraph of $G \setminus K'$ that is neither of type branch nor of type triangle with respect to K' . By Lemma 4.5, P is an augmenting path for K' . Since

the attachment of p over K is included in a flat branch of K , as is the attachment of p' over K , and since the interior of P is strongly anti-complete to K , we deduce that P is an augmenting path for K in G . This completes the argument. \square

Proposition 4.8. *Let H be a chordless subdivision of a 3-connected graph H_0 , and let $E \subseteq E(H)$. Assume that the edges in E do not all belong to the same flat branch of H , and that some two edges of E are vertex-disjoint. Then H contains an induced subgraph K such that K is a chordless subdivision of K_4 , and such that there exist vertex-disjoint edges $ab, cd \in E \cap E(K)$ that do not belong to the same flat branch of K .*

Proof. We first observe that each edge of H_0 was subdivided at least once to obtain H . For suppose that some edge uv of H_0 remained unsubdivided in H . Since H_0 is 3-connected, Menger's theorem guarantees that there are three internally disjoint paths between u and v in H_0 . At least two of those paths do not use the edge uv , and by putting them together, we obtain a cycle Z_0 of H_0 ; clearly, the edge uv is a chord of Z_0 in H_0 . Now, a subdivision Z of Z_0 is a cycle of H , and since uv remained unsubdivided in H , we see that uv is a chord of the cycle Z in H , contrary to the fact that H is chordless. Thus, no edge of H_0 remained unsubdivided in H . Note that this implies that H is triangle-free, and consequently, that all branches of H are flat.

Next, note that the branch vertices of H are precisely the vertices of H_0 . Furthermore, the flat branches of H are precisely the paths of H that were obtained by subdividing the edges of H_0 . This implies that every edge of H belongs to a unique flat branch of H , and it also implies that no two distinct flat branches of H share more than one vertex. Finally, the branch vertices of H (that is, the vertices of H_0) all belong to more than one flat branch of H .

We now claim that there exist vertex-disjoint edges $ab, cd \in E$ such that ab and cd do not belong to the same flat branch of H . By hypothesis, there exist vertex-disjoint edges $ab, a'b' \in E$. If ab and $a'b'$ do not belong to the same flat branch of H , then we set $c = a'$ and $d = b'$, and we are done. Suppose now that ab and $a'b'$ do belong to the same flat branch (call it F) of H . Since edges in E do not all belong to the same flat branch of H , there exists an edge $cd \in E$ such that cd does not belong to the flat branch F . Since every edge of H belongs to a unique flat branch of H , and since two distinct flat branches of H have at most one vertex in common, this implies that cd is vertex-disjoint from at least one of ab and $a'b'$; by symmetry, we may assume that cd is vertex-disjoint from ab . This proves our claim.

So far, we have found vertex-disjoint edges $ab, cd \in E$ such that ab and cd do not belong to the same flat branch of H . Since H is a subdivision of a 3-connected graph, we know that H contains a cycle Z such that $ab, cd \in$

$E(Z)$; since H is chordless, the cycle Z of H is induced. By symmetry, we may assume that the vertices a, b, c, d appear in this order in Z . Since H is a subdivision of a 3-connected graph, Lemma 2.2 implies that H is cyclically 3-connected. Let P_{ad} be the subpath of Z such that the endpoints of P_{ad} are a and d , and $b, c \notin V(P_{ad})$, and let P_{bc} be defined in an analogous fashion. By Lemma 2.4, there exists a path R of H such that one endpoint of R belongs to P_{ad} , the other endpoint of R belongs to P_{bc} , no internal vertex of R belongs to Z , and R is not between a and b or between c and d . Since H is chordless, the path R is of length at least two. Clearly, $Z \cup R$ is a theta, and since H is chordless, this theta is an induced subgraph of H . Further, since H is a subdivision of a 3-connected graph, we know that H is not a theta, and so Lemma 2.1 implies that there exists a subgraph K of H such that K is a subdivision of K_4 , and the theta $Z \cup R$ is a subgraph of K ; since H is chordless, we know that K is in fact a chordless subdivision of K_4 and an induced subgraph of H , and that the theta $Z \cup R$ is an induced subgraph of K . Since the edges ab and cd do not belong to the same flat branch of the theta $Z \cup R$, we know that ab and cd do not belong to the same flat branch of K . This completes the argument. \square

Lemma 4.9. *Let G be an $\{ISK_4, \text{wheel}, \text{diamond}\}$ -free trigraph, let K be an induced subtrigraph of G such that K is a line trigraph of a cyclically 3-connected, chordless graph H of maximum degree at most three, and let P be an inclusion-wise minimal connected induced subtrigraph of $G \setminus V(K)$ such that P is neither of type triangle nor of type branch with respect to K . Then P is an augmenting path for K in G .*

Proof. We first observe that P is a non-null trigraph. Indeed, since H is cyclically 3-connected, H contains a vertex of degree at least three (in fact, since the maximum degree of H is at most three, we know that H contains a vertex of degree exactly three). Since K is a line trigraph of H , it follows that K contains a strong triangle. Clearly, the (empty) attachment of the null trigraph over K in G is included in this strong triangle of K , and so since P is not of type triangle, we see that P is non-null.

If H is a theta, then K is prism, and the result follows immediately from Lemma 4.5. So assume that H is not a theta; then by Lemma 2.2, H is a subdivision of a 3-connected graph. Since H is chordless, we know that H is in fact a chordless subdivision of a 3-connected graph. Now, since P is neither of type triangle nor of type branch with respect to K , we know that some two edges of H (equivalently: vertices of K) in the attachment of P over K are vertex-disjoint. Further, since P is not of type branch with respect to K , we know that the edges of H (equivalently: vertices of K) in the attachment of P over K do not all belong to the same flat branch of H . Let E be the attachment of P over K . Proposition 4.8 now implies that H contains an induced subgraph H' such that H' is a chordless subdivision of

K_4 , and such that there exist vertex-disjoint edges $ab, cd \in E \cap E(H')$ that do not belong to the same flat branch of H' . Let $K' = K[E(H')]$. Since the vertices ab and cd of K belong to the attachment of P over K' , we easily deduce that P is a minimal connected induced subtrigraph of $G \setminus V(K')$ such that P is neither of type triangle nor of type branch with respect to K' . By Lemma 4.7 then, we know that P is an augmenting path for K' .

If $K = K'$, then P is an augmenting path for K , and we are done. So assume that $V(K') \subsetneq V(K)$. Let A be the attachment of P over K' . Since P is an augmenting path for K' , we know that $|A| = 4$, and furthermore, there exists a cycle Z' of H' (since H' is chordless, the cycle Z' is induced) such that $A \subseteq E(Z')$. Now, we claim that A is the attachment of P over K . Suppose otherwise. Fix $xy \in E(H) \setminus A$ such that xy is in the attachment of P over K . We now apply Lemma 2.5 to the graph H , the cycle Z' , and the edge xy , and we deduce that H contains a subdivision H'' of K_4 that contains Z' and xy ; since H is chordless, so is H'' . Set $K'' = K[E(H'')] = G[E(H'')]$. Since the attachment of P over K'' contains at least five vertices (because it includes $A \cup \{xy\}$), we know that P is not of type triangle with respect to K . Further, because $A \subseteq E(Z')$, and $xy \notin E(Z')$, we deduce that $A \cup \{xy\}$ is not included in a flat branch of $K[E(H'')]$, and so P is not of type branch. We now deduce that P is a minimal connected induced subgraph of $G \setminus V(K'')$ such that P is neither of type branch nor of type triangle with respect to K'' (the minimality of P with respect to K'' follows from the minimality of P with respect to K). Thus, by Lemma 4.7, P is an augmenting path for K'' . But this is impossible because the attachment of P over K'' contains at least five vertices. Thus, the attachment of P over K is precisely A .

So far, we have shown that P is an augmenting path for K' , and that the attachment of P over K is the same as the attachment of P over K' . In order to show that P is augmenting path for K , it now only remains to show that neither of the two strongly adjacent pairs of K that belong to the attachment of P over K belongs to any triangle of K . But this follows immediately from the fact that G is diamond-free. \square

Proposition 4.10. *Let G be an $\{ISK_4, \text{wheel}, \text{diamond}\}$ -free trigraph, let K be an inclusion-wise maximal induced subtrigraph of G such that K is a line trigraph of a cyclically 3-connected, chordless graph H of maximum degree at most three. Then every component of $G \setminus V(K)$ is either of type branch or of type triangle with respect to K .*

Proof. We prove a slightly stronger statement: every connected induced subtrigraph of $G \setminus V(K)$ is either of type branch or of type triangle with respect to K . Suppose otherwise, and let P be a minimal induced subtrigraph of $G \setminus V(K)$ that is neither of type branch nor of type triangle with respect to K . Then by Lemma 4.9, P is an augmenting path for K in G . Now we show that $G[V(K) \cup V(P)]$ is a line trigraph of a cyclically 3-connected, chordless

graph H of maximum degree at most three (this will contradict the maximality of K). Since K is a line trigraph, we know that every triangle of K is strong, and since P is an augmenting path for K , we also know that the two triangles formed by the endpoints of P and their attachments over K are strong. It follows that all triangles of $G[V(K) \cup V(P)]$ are strong, and we need only show that the full realization of $G[V(K) \cup V(P)]$ is the line graph of a cyclically 3-connected, chordless graph H of maximum degree at most three. Thus, we may assume that $G[V(K) \cup V(P)]$ contains no semi-adjacent pairs, that is, that $G[V(K) \cup V(P)]$ is a graph. Let p and p' be the endpoints of P .

Let $\{ab, bc\} \subseteq E(H) = V(K)$ be the attachment of p over K , and let $\{a'b', b'c'\} \subseteq E(H) = V(K)$ be the attachment of p' over K . Since p is of type branch with respect to K (because P is an augmenting path for K), we know that the vertices ab and bc of K belong to the same flat branch of K , and consequently, that the vertex b (of H) is of degree two in H . Similarly, the vertex b' (of H) is of degree two in H . Let H' be the graph obtained by adding to H a path R between b and b' of length one more than the length of P . It is then easy to see that $G[V(K) \cup V(P)]$ is the line graph of H' . The fact that H' is cyclically 3-connected follows from Lemma 2.3, and the fact that H' is chordless and of maximum degree at most three follows from the fact that its line graph is $\{\text{ISK4}, \text{wheel}\}$ -free. Thus, $G[V(K) \cup V(P)]$ contradicts the maximality of K . This completes the argument. \square

Lemma 4.11. *Let G be an $\{\text{ISK4}, \text{wheel}, \text{diamond}\}$ -free trigraph such that some induced subtrigraph of G is a line trigraph of a cyclically 3-connected, chordless graph of maximum degree at most three. Then either G is a line trigraph of a cyclically 3-connected, chordless graph of maximum degree at most three, or G admits a clique-cutset or a stable 2-cutset.*

Proof. Let K be a maximal induced subgraph of G that is a line trigraph of a cyclically 3-connected, chordless graph of maximum degree at most three. If $G = K$, then we are done. So assume that $V(K) \subsetneq V(G)$. We may also assume that G does not admit a clique-cutset, for otherwise we are done. (In particular then, G is connected, and G does not contain a cut-vertex.) Note that this implies that no component of $G \setminus V(K)$ is of type triangle with respect to K , for otherwise, the attachment of such a component over K would be a clique-cutset of G . Proposition 4.10 now guarantees that all components of $G \setminus V(K)$ are of type branch with respect to K . Since $V(K) \subsetneq V(G)$, and since every component of $G \setminus V(K)$ is of type branch with respect to K , we know that there exists some flat branch B of K such that the attachment of some component of $G \setminus V(K)$ is included in $V(B)$. Let b and b' be the endpoints of B . Then $\{b, b'\}$ is a cutset of G . If bb' were a strongly adjacent pair, then $\{b, b'\}$ would be a clique-cutset of G , contrary to the fact that G admits no clique-cutset. So bb' is an anti-adjacent pair, and it follows that $\{b, b'\}$ is a stable 2-cutset. \square

4.4 Decomposing $\{\text{ISK4}, \text{wheel}\}$ -free graphs

We remind the reader that a trigraph is *series-parallel* if its full realization is series-parallel. We also remind the reader that a *complete bipartite trigraph* is a trigraph whose vertex set can be partitioned into two strongly stable sets, strongly complete to each other; if these two strongly stable sets are each of size at least three, then the complete bipartite trigraph is said to be *thick*. A trigraph is a *strong $K_{3,3}$* if its full realization is a $K_{3,3}$ and it contains no semi-adjacent pairs. (Clearly, a strong $K_{3,3}$ is a thick complete bipartite trigraph.)

Proposition 4.12. *Let G be an $\{\text{ISK4}, \text{wheel}\}$ -free trigraph whose full realization contains a prism as an induced subgraph. Then G contains a prism as an induced subtrigraph.*

Proof. Let H be an induced subtrigraph of G such that the full realization of H is a prism. Then the two triangles of H are strong, for otherwise, H would be an ISK4-trigraph, contrary to the fact that G is ISK4-free. It follows that the trigraph H is a prism. \square

Proposition 4.13. *Let G be an ISK4-free $K_{3,3}$ -trigraph. Then G contains no semi-adjacent pairs (i.e., G is a strong $K_{3,3}$).*

Proof. Since G is a $K_{3,3}$ -trigraph, we know that $V(G)$ can be partitioned into two stable sets of size three, say $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, that are complete to each other. Note that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are strongly complete to each other, for otherwise, G would be an ISK4-trigraph, contrary to the fact that G is ISK4-free. Further, $\{a_1, a_2, a_3\}$ is a strongly stable set, for otherwise, $G \setminus b_3$ would be an ISK4-trigraph; similarly, $\{b_1, b_2, b_3\}$ is a strongly stable set. Thus, G contains no semi-adjacent pairs, and G is a strong $K_{3,3}$. \square

The following is Lemma 2.2 from [3].

Lemma 4.14. [3] *Let G be an ISK4-free graph. Then either G is a series-parallel graph, or G contains a prism, a wheel, or a $K_{3,3}$ as an induced subgraph.*

Lemma 4.15. *Let G be an $\{\text{ISK4}, \text{wheel}\}$ -free trigraph. Then either G is series-parallel, or G contains a prism or a strong $K_{3,3}$ as an induced subtrigraph.*

Proof. Let \tilde{G} be the full realization of G . Then \tilde{G} is an $\{\text{ISK4}, \text{wheel}\}$ -free graph, and so by Lemma 4.14, we know that either \tilde{G} is a series-parallel graph, or \tilde{G} contains a prism or a $K_{3,3}$ as an induced subgraph. In the former case, G is a series-parallel trigraph, and we are done. So assume that \tilde{G} contains a prism or a $K_{3,3}$ as an induced subgraph. Then by Propositions 4.12 and 4.13, G contains a prism or a strong $K_{3,3}$ as an induced subtrigraph. \square

Proposition 4.16. *Let G be an $\{ISK4, \text{wheel}\}$ -free trigraph, let H be a maximal thick bipartite induced subtrigraph of G , let (A, B) be a bipartition of H , and let $v \in V(G) \setminus V(H)$. Then v has at most one neighbor in A and at most one neighbor in B .*

Proof. Suppose first that v has at least two neighbors in each of A and B . Then there exist distinct $a_1, a_2 \in A$ and distinct $b_1, b_2 \in B$ such that v is complete to $\{a_1, a_2, b_1, b_2\}$; but now $G[v, a_1, a_2, b_1, b_2]$ is a wheel-trigraph, contrary to the fact that G is wheel-free.

Exploiting symmetry, we may now assume that v has at most one neighbor in B ; then v has at least two strong anti-neighbors in B , call them b_1 and b_2 . If v has at most one neighbor in A , then we are done. So assume that v has at least two neighbors in A , call them a_1 and a_2 . Suppose first that v has at least one strong anti-neighbor in A , call it a . Then $G[v, a_1, a_2, a, b_1, b_2]$ is an $ISK4$ -trigraph, contrary to the fact that G is $ISK4$ -free. Thus, v is complete to A . Suppose first that v has a weak neighbor in A , and let $A' \subseteq A$ be such that v has a weak neighbor in A' , and $|A'| = 3$. But then $G[A' \cup \{v, b_1, b_2\}]$ is a $K_{3,3}$ -trigraph that contains a semi-adjacent pair, contrary to Proposition 4.13. Thus, v is strongly complete to A . If v is strongly anti-complete to B , then $G[V(H) \cup \{v\}]$ contradicts the maximality of H . It follows that v has a unique neighbor in B , call it b . But then v, b, b_1 , and any two vertices of A induce an $ISK4$ -trigraph in G , contrary to the fact that G is $ISK4$ -free. \square

Proposition 4.17. *Let G be an $\{ISK4, \text{wheel}\}$ -free trigraph, let H be a maximal induced thick bipartite subtrigraph of G , let (A, B) be a bipartition of H , and let C be a component of $G \setminus V(H)$. Then the attachment of C over H contains at most one vertex of A and at most one vertex of B .*

Proof. Suppose otherwise, and fix a minimal connected induced subtrigraph P of C such that the attachment of P over H either contains more than one vertex of A , or more than one vertex of B . By Proposition 4.16, P has at least two vertices.

Let us first show that P is a narrow path. Suppose otherwise. By symmetry, we may assume that the attachment of P over H contains at least two vertices of A , call them a_1 and a_2 . For each $i \in \{1, 2\}$, let P_i be the set of all vertices of P that are adjacent to a_i . By construction, P_1 and P_2 are non-null, and by Proposition 4.16, they are disjoint. Using the fact that P is connected, we let P' be a narrow path in P whose one endpoint belongs to P_1 , and whose other endpoint belongs to P_2 . Since P is not a narrow path, we know that $P' \neq P$. But now P' contradicts the minimality of P . This proves that P is a narrow path. Let p and p' be the endpoints of P . (Since $|V(P)| \geq 2$, we know that $p \neq p'$.)

By symmetry, we may assume that the attachment of P over H contains at least two vertices of A . By the minimality of P , each end-vertex of P has

a neighbor in A ; by Proposition 4.16 then, each endpoint of P has a unique neighbor in A . Let a be the unique neighbor of p in A , and let a' be the unique neighbor of p' in A . By the minimality of P , we know that $a \neq a'$, and that the interior of P is strongly anti-complete to A . In particular then, the attachment of P over A contains exactly two vertices.

By the minimality of P , we know that the attachment of P over H contains at most two vertices of B . Now, suppose that the attachment of P over H contains exactly two vertices of B , call them b and b' . Then by the minimality of P , we know that one of p and p' is adjacent to b (and strongly anti-adjacent to b'), the other one is adjacent to b' (and strongly anti-adjacent to b), and the interior of P is strongly anti-complete to B . By symmetry, we may assume that p is adjacent to b and strongly anti-adjacent to b' , and that p' is adjacent to b' and strongly anti-adjacent to b . Using the fact that $|B| \geq 3$, we fix some $b'' \in B \setminus \{b, b'\}$. But then $G[V(P) \cup \{a, a', b, b''\}]$ is an ISK4-trigraph, contrary to the fact that G is ISK4-free.

From now on, we assume that the attachment of P over H contains at most one vertex of B . Since $|B| \geq 3$, it follows that there exist two distinct vertices in B (call them b_1 , and b_2) that are strongly anti-complete to P . Further, since $\{a, a'\}$ is the attachment of P over A , and since $|A| \geq 3$, there exists some $a'' \in A$ that is strongly anti-complete to P . But now $G[V(P) \cup \{a, a', a'', b_1, b_2\}]$ is an ISK4-trigraph, contrary to the fact that G is ISK4-free. \square

Lemma 4.18. *Let G be an $\{\text{ISK4}, \text{wheel}\}$ -free trigraph that contains a strong $K_{3,3}$ as an induced subtrigraph. Then either G is a thick complete bipartite trigraph, or G admits a clique-cutset.*

Proof. Let H be a maximal induced subtrigraph of G such that H is a thick bipartite trigraph (such an H exists because G contains a strong $K_{3,3}$ as an induced subtrigraph). If $H = G$, then G is a thick complete bipartite trigraph, and we are done. So assume that $H \neq G$. Let (A, B) be a bipartition of H . Fix a component C of $G \setminus V(H)$. Then by Proposition 4.17, the attachment of C over H contains at most one vertex of A , and at most one vertex of B . Since A is strongly complete to B , it follows that the attachment of C over H is a clique-cutset of G . \square

4.5 Proof of the main theorem

We are finally ready to prove Theorem 4.1, our decomposition theorem for $\{\text{ISK4}, \text{wheel}\}$ -free trigraphs. In fact, we prove a slightly stronger theorem (Theorem 4.19), stated below. Clearly, Theorem 4.1 is an immediate corollary of Theorem 4.19.

Theorem 4.19. *Let G be an $\{\text{ISK4}, \text{wheel}\}$ -free trigraph. Then at least one of the following holds:*

- G is a series-parallel trigraph;
- G is a thick complete bipartite trigraph;
- G is a line trigraph of a cyclically 3-connected, chordless graph of maximum degree at most three;
- G admits a clique-cutset;
- G admits a stable 2-cutset.

Proof. We may assume that G is diamond-free, for otherwise, Proposition 4.3 guarantees that G admits a clique-cutset or a stable 2-cutset, and we are done. Further, by Lemma 4.15, either G is a series-parallel trigraph, or it contains a prism or a strong $K_{3,3}$ as an induced subtrigraph. If G is series-parallel, then we are done. If G contains a strong $K_{3,3}$ as an induced subtrigraph, then by Lemma 4.18, either G is a thick complete bipartite trigraph, or G admits a clique-cutset, and in either case, we are done. It remains to consider the case when G contains an induced prism. Note that the prism is a line trigraph of a theta, and a theta is a cyclically 3-connected, chordless graph of maximum degree at most three. Now let K be a maximal induced subtrigraph of G such that K is a line trigraph of a cyclically 3-connected, chordless graph of maximum degree at most three. If $K = G$, then we are done. Otherwise, Lemma 4.11 guarantees that G admits a clique-cutset or a stable 2-cutset. This completes the argument. \square

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